Hamiltonian Reduction & GIT

[Kirillov]

$G \curvearrowright M$ $\qquad$ G - real Lie group

$\qquad\qquad\qquad\quad$ M - C^∞-mfd

the action is proper if the map

$$(m,g) \longmapsto (m, g\cdot m) \text{ is}$$

$\Longleftrightarrow$ $\{g \in G \mid gK_1 \cap gK_2 \neq \emptyset\} \subset G$ is compact for compact $K_1, K_2 \subset G$

For proper gp actions we have a good local model

Thm: M - C^∞ mfd w/ proper Lie gp actn G
$x \in G$, $\exists$ (locally closed) submfd $S \ni x$ invariant under the action of the stabilizer G_x & G-invi nbhd $U \supset O_x$ (orbit) st

$$G \times_{G_x} S \longrightarrow U \text{ is an isomorphism}$$

[Lina Slice Chicago REU Paper]

<u>Def</u>: (Affine) morphism of varieties means a function btwn varieties st it's polynomial in each coord.

<u>Def</u>: A rational mapping $f: X \to Y$ is an equiv class (f_Z, Z), $Z \subset X$ nonempty $f_Z : Z \to Y$ st $(f_Z, Z) \sim (f_{Z'}, Z')$ if

$$f_Z\big|_{Z \cap Z'} = f_{Z'}\big|_{Z \cap Z'}$$

it's birational if $\exists g: Y \to X$ also rational

<u>Def</u>: G - alg gp, V fin dim v.s.
a morphism of gps $G \longrightarrow GL(V)$
is called a rational representation of G
if it's a rational map of varieties

<u>Def</u>: G is reducible if it contains a nontrivial G-inv. subspace

Def: G is linearly reductive if every rational rep of G is completely reducible

Note: In char 0

Reductive $\iff$ Radical of G is a torus

Categorical & Good Quotients

Def: If G is an alg. gp & X a variety acted on by G, the categorical quotient is a pair (Y, π), Y - variety, π is a G-invariant morphism $X \to Y$ st $\forall$ Z variety, if $f: X \to Z$ is G-inv. morphism $\exists!$ $\bar{f}: Y \to Z$ st

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowleft"] & Z \\ Y \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

<u>Def</u>: X - variety, $G \curvearrowright X$ alg gp

A good quotient is (Y, π), Y - variety

$\pi : X \to Y$ morphism st

π is affine, surjective & G-invariant

$A(X)$ ring of reg. functions on X

If U open in Y, $A(U) \xrightarrow{\sim} A(\pi^{-1}(U))^G$

is omorphism

If $V_1, V_2 \subset X$ disjoint, closed, G-invariant

$\pi(V_1) \cap \pi(V_2) = \emptyset$

We denote this quotient $Y = X /\!/ G$

<u>Thm</u>: X - affine variety, G-reductive gp acting on X

$X/\!/G$ exists & is affine

[Kirillov]

$$M /\!/ G = \mathrm{Spec}\left(\underline{k[M]^G}\right)$$

↳ algebra of G-inv. polynomial functions on M &

Spec is the set of max ideals

(not prime?)

G - reductive, M - affine

$M/G \longrightarrow M/\!/G$ is surjective &

$\mathcal{O}_x, \mathcal{O}_{x'}$ define the same pt iff

$$\overline{\mathcal{O}_x} \cap \overline{\mathcal{O}_{x'}} \neq \emptyset$$

Thm: $M/\!/G = \{\text{closed orbits in } M\}$

$[x] \mapsto$ unique closed orbit contained in $\overline{\mathcal{O}_x}$

Ex) $M = \mathbb{A}^1$, $G = k^\times$ w/ action
$$\lambda(x) = \lambda \cdot x$$

There are two orbits $\mathcal{O}_1 = \{0\}$
$\mathcal{O}_2 = k^\times = M \setminus \{0\}$, so $M/\!/G$ has 2 pts
set theoretically, but only one of these is closed

$$\overline{\mathcal{O}_1} = \mathcal{O}_1, \quad \overline{\mathcal{O}_2} = \mathcal{O}_2 \cup \mathcal{O}_1 = M$$

So $$\mathbb{A}^1 /\!/ k^\times = \{pt\}$$
Indeed $$k[\mathbb{A}^1]^G = k[x]^{k^\times} = k$$

Ex) $$M = \mathrm{End}(k^n)$$
we can always change basis to upper Δ
so any $Gl(n)$-inv. poly is completely
determined by its diagonal values so

$$k[M]^{Gl_n} = k[\lambda_1, \dots, \lambda_n]^{S_n}$$

$$M /\!/ GL(n) = \mathrm{Spec}\left(k[\lambda_1, \dots, \lambda_n]^{S_n}\right)$$
$$= k^n / S_n \simeq k^n$$

If we wish to find out info regarding nonclosed orbits we need to investigate projective varieties

For a projective variety $X \subset \mathbb{P}^r$, there is a graded algebra

$$A = \bigoplus_{n \geq 0} A_n$$

A_n = homogeneous polys of deg n restr. to X

We can recover X from this via

$$X = \mathrm{Proj}(A)$$

Where $\mathrm{Proj}(A)$ = graded ideals $J \subset A$ maximal among graded ideals not containing $A_+ = \bigoplus_{n>0} A_n$

If $A = \bigoplus_{n \geq 0} A_n$ is fin generated graded w/o nilpotents, we can define the quasi-projective variety $X = \mathrm{Proj}(A)$

there is a natural morphism

$$\begin{tikzcd} X = \mathrm{Proj}(A) \arrow[d, "\pi"] \\ X_0 = \mathrm{Spec}(A_0) \end{tikzcd}$$

this is projective, each fiber $\pi^{-1}(x)$ is a proj variety

Let χ be a character of G, i.e.

$$\chi : G \to k^\times$$

Define

$$k[M]^{G,\chi} = \{ f \in k[M] \mid f(g.m) = \chi(g) f(m) \}$$

semi-invariants

We can define the corresponding quasiproj. variety

$$M /\!/_{\chi} G = \mathrm{Proj}\left(\bigoplus_{n \geq 0} k[M]^{G, \chi^n} \right)$$

twisted GIT quotient

Def: χ - character of G, lift the action of G on M to an action on $M \times k$ by

$$g(m, z) = (g(m), \chi^{-1}(g) z)$$

A pt $x \in M$ is called semi-stable if $\forall z \in k^{\times}$

$$\overline{\mathcal{O}_{(x,z)}} \cap (\underbrace{M \times \{0\}}_{\text{zero section}}) = \emptyset$$

the set of semi-stable pts is M^{ss}_{χ}

Thm: $x \in M$ is χ-ss iff $\exists f \in k[M]^{G, \chi^n}$ $n \geq 1$ st $f(x) \neq 0$

Thm: $M//_\chi G = \{\text{closed orbits in } M^{ss}\}$

as a top space

Def: $x \in M^{ss}$ is stable if G_x is finite & $\mathcal{O}_x$ is closed in M^{ss}

$$M^s/G \subset M//_\chi G$$

Ex $M = \mathbb{A}^2$, $G = k^\times$ $G \curvearrowright M$ by multiplication

$t \cdot (x_1, x_2) = (tx_1, tx_2)$

$M/G = \{0\} \cup \mathbb{P}^1(k)$

$M//G = \{pt\}$, if $\chi(\lambda) = \lambda$

$M^{ss} = \mathbb{A}^2 - \{0\} = M^s$ & $M//_\chi G = \mathbb{P}^1(k)$

$M//_\chi G = \mathrm{Proj}\left(\bigoplus k[\mathbb{A}^2]^{G, \chi^n}\right)$

$$= \mathrm{Proj}\left(\bigoplus_{n\geq 0}\{f\in k[\mathbb{A}^2] \mid f(g\cdot m) = \chi^n(g) f(m)\}\right)$$

$$= \mathrm{Proj}\left(\bigoplus_{n\geq 0}\{f\in \underset{\mathbb{C}[x_1,x_2]}{\overset{\shortparallel}{k[\mathbb{A}^2]}} \mid f(g\cdot m) = g^n f(m)\}\right)$$

$$= \mathrm{Proj}\left(\bigoplus_{n\geq 0}\{f\in \mathbb{C}[x_1,x_2] \mid \underbrace{f(g\cdot(x_1,x_2)) = g^n f(x_1,x_2)}\}\right)$$

$$= \mathrm{Proj}\left(\bigoplus_{n\geq 0} \mathbb{C}[x_1,x_2]^n\right)$$

$$= \mathbb{P}^1$$

we can always factor out constants

Same for $\chi(\lambda) = \lambda^n$

but if $\chi(\lambda) = \lambda^{-n}$

$M^{ss} = \emptyset$] Can't reconcile non-neg & pos powers

Def: $x \in M$ is regular if the orbit $\mathcal{O}_x$ is closed & the stabilizer $G_x = 1$

Ex

$$M = \{ (i,j) \mid j : k \to k^2,\ i : k^2 \to k,\ ij = 0 \}$$

$$k \underset{i}{\overset{j}{\rightleftarrows}} k^2$$

$k^\times \curvearrowright M$ $\qquad \lambda \cdot (i,j) = (\lambda i, \lambda^{-1} j)$

$A = ji : k^2 \to k^2$ are $k^\times$ invariant so

$$M /\!/ k^\times \longrightarrow M_{2\times 2}(k)$$

$$(i,j) \longmapsto ji$$

the image of this is the variety

$$Q = \{ A \in M_{2\times 2}(k) \mid tr(A) = det(A) = 0 \}$$

χ is identity

$$M /\!/_\chi k^\times = \left\{ (V, i) \;\middle|\; V \subset k^2,\ \begin{array}{l} \dim V = 1 \\ i : k^2 \to V \\ i|_V = 0 \end{array} \right\}$$

$$M /\!/_\chi k^\times \to \mathbb{P}^1(k)$$

$\uparrow$ line bundle over $\mathbb{P}^1$

$$M \simeq T^*\mathbb{P}^1$$

[Kirillov]

Symplectic Geometry

Def: A Poisson structure on a mfd X is a k-bilinear morphism of structure sheaves

$$\{\ ,\ \} : \mathcal{O}_X \times \mathcal{O}_X \to \mathcal{O}_X$$

(i) Skew-symmetric $\{f, g\} = -\{g, f\}$

(ii) Jacobi identity

(iii) $\{f, gh\} = \{f, g\} h + g \{f, h\}$

$\varphi : M \to N$ is called a Poisson morphism if

$$\varphi^*\{f,g\} = \{\varphi^* f, \varphi^* g\}$$

Ex) Let $\mathfrak{g}$ fin dim Lie alg $/ k$

$\mathfrak{g}^*$ considered as a mfd $/ k$ has

a Poisson structure

$$\{x,y\} = [x,y]$$

Def: A symplectic mfd M is a mfd w/

a closed non-degen. 2-form $\omega \in \Omega^2(M)$

$$T_x M \longrightarrow T_x^* M$$
$$\xi \longmapsto \omega(-,\xi)$$

$\forall f \in \mathcal{O}_M$, $\exists!$ X_f vect. field

s.t. $\omega(-, X_f) = df$

Lemma: M - symplectic, $f, g \in C_M$

$$\{f, g\} = \omega(X_g, X_f)$$

is a Poisson bracket

Ex) $T^*X = \{(x, \lambda) : x \in X, \lambda \in T_x^* X\}$

$$\langle \alpha, v \rangle = \langle \lambda, \pi_* v \rangle$$

$$v \in T_{(x,\lambda)}(T^*X)$$

$$\pi : T^*X \to X$$

$$\pi_* v \in T_x X$$

Thm: T^*X is symplectic where $\omega = d\alpha$

In local coord, if q^i are local coord on X

p_i are coord. on T_x^*X, then

p_i, q^i are local coord on T^*X

$$\omega = \sum dp_i \wedge dq^i$$

$$\{f,g\} = \sum_i \left(\frac{\partial g}{\partial p_i}\frac{\partial f}{\partial q^i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i} \right)$$

Ex) V – fin dim vect. space/$\mathbb{R}$

$$T^*V = V \oplus V^*$$

$$\omega((v_1,\lambda_1),(v_2,\lambda_2)) = \langle \lambda_1, v_2 \rangle - \langle \lambda_2, v_1 \rangle$$

Ex) $Y \subset X$ submfd

N^*Y conormal bundle to N:

$$N^*_x Y = (T_x X / T_x Y)^*$$

M - symplectic, $\quad G \curvearrowright M$

$\forall a \in \mathfrak{g} = \mathrm{Lie}(G)$ defines a vect. field ξ_a

$\mathcal{L}_{\xi_a}(\omega) = 0 \qquad X_{H_a} = \xi_a = \frac{d}{dt}\left(\exp(t\xi)_a\right)\Big|_{t=0}$

Get

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\xi_a"] \arrow[dr, dashed, "\mu^*"'] & \mathrm{Vect}(M) \\ & \mathrm{Func}(M) \arrow[u] \end{tikzcd}$$

$f \mapsto X_f \mapsto \omega(-, X_f)$

<u>Def</u>: A comoment map is a map

$$\mu^* : \mathfrak{g} \longrightarrow \mathrm{Func}(M)$$

st (1) Diagram commutes

(2) μ^* is a Lie alg. hom

<u>Def</u>: $\mu : M \to \mathfrak{g}^*$ Moment map

$$\mu(p) : \left(\xi \longmapsto \mu^*(\xi)(p) \right)$$

$$\langle \mu(p), \xi \rangle = \mu^*(\xi)(p)$$

Prop: μ^* Lie alg hom $\Leftrightarrow$ μ is G-equiv.

Ex) $G \curvearrowright X \rightsquigarrow G \curvearrowright T^*X$

$$\langle \mu(x, \lambda), a \rangle = \langle \lambda, \xi_a(x) \rangle$$

$a \in \mathfrak{g}$

$\lambda \in T_x^*X$

$x \in X$

Ex) $T^*\mathbb{R} = \mathbb{R} \times \mathbb{R}$ [Hunter's notes]

$\overset{\shortparallel}{X}$ $\quad q \quad p$

$G = \mathbb{R} \curvearrowright X$ by translations

$g \mapsto (q \mapsto q+g)$

$\omega = dq \wedge dp$

$G \curvearrowright T^*\mathbb{R}$

$g \mapsto ((q,p) \mapsto (q+g, p))$

- $\omega : \mathrm{Vect}(T^*\mathbb{R}) \longrightarrow \Omega^1(T^*\mathbb{R})$

$$a\partial_q + b\partial_p \longmapsto (c\partial_q + d\partial_p \mapsto \omega(a\partial_q + b\partial_p, c\partial_q + d\partial_p)$$
$$= dq \wedge dp(-,-)$$
$$= ad - bc$$

- $f \in \mathrm{Func}(T^*\mathbb{R}) \longrightarrow \Omega^1(T^*\mathbb{R}) \xrightarrow{\omega^{-1}} \mathrm{Vect}(T^*\mathbb{R})$

$$f \longmapsto \frac{\partial f}{\partial q} dq + \frac{\partial f}{\partial p} dp \mapsto \frac{\partial f}{\partial p} \partial_p - \frac{\partial f}{\partial q} \partial_q$$

As a vect. field: $\dfrac{\partial f}{\partial p} \partial_q - \dfrac{\partial f}{\partial q} \partial_p$

- Infinetesimal action

$$\xi \in \mathfrak{g} = \mathbb{R}$$

$$\vec{\xi}_q = \frac{d}{dt}\left(\exp(t\xi)\cdot q\right)\Big|_{t=0}$$

$$= \frac{d}{dt}\left(q + \exp(t\xi)\right)\Big|_{t=0}$$

$$= \xi\,\partial_q$$

-

$$\begin{tikzcd}
\mathfrak{g} \arrow[r] \arrow[dr, dashed] & \mathrm{Vect}(T^*\mathbb{R}) = \left\{ a\frac{\partial}{\partial q} + b\frac{\partial}{\partial p} \right\} \\
& \mathrm{Func}(T^*\mathbb{R}) = \{ f(q,p) \} \arrow[u]
\end{tikzcd}$$

$$\xi \mapsto \xi\,\partial_q$$

$$\xi \mapsto \xi\cdot p$$

$$f \mapsto f_p\,\partial_q - f_q\,\partial_p$$

- Poisson bracket

$$\{f,g\} = f_q\,g_p - f_p\,g_q$$

- μ^* Lie hom

$$\mu^*([\xi_1, \xi_2]) = \mu^*(0) = 0$$

- $\xi_p = \mu^*(\xi)(q, p) = \langle \mu(q,p), \xi \rangle$

$$= \langle p\,dq, \xi\,dq \rangle$$

in dual basis $= p\,\xi$

$\mathfrak{g}^* \cong \mathbb{R}$

$\mu(q,p) = p$

Ex) $L = \mathrm{Hom}(V, W)$

$$M = T^*L = L \oplus L^*$$

$$G = GL(V) \curvearrowright L$$

$$g \cdot u = ug^{-1} \quad u \in L$$

$$\mu : L \oplus L^* \longrightarrow \mathfrak{gl}(V)$$
$$(u, v) \longmapsto -vu$$

If $\underbrace{G}_{\text{gp}} \curvearrowright \underbrace{M}_{\text{Symplectic mfd}}$ is Hamiltonian ($\exists$ moment map)

We can create a construction to guarantee $M//G$ is Symplectic

Thm: M be C^∞-symplectic w/ proper Hamiltonian action of real Lie group G;

$\mu : M \to \mathfrak{g}^*$ moment map

Let $p \in \mathfrak{g}^*$ be st

(1) $\mu^{-1}(p) \subset M$ is a submfd

(2) $\mu^{-1}(p)/G_p$ is smooth

then $\mu^{-1}(p)/G_p$ is symplectic

The most interesting case is when X is C^∞ $M = T^*X$, then,

Thm: $\mu : T^*X \to \mathfrak{g}^*$ moment map

$\mu^{-1}(0)/G$ has structure of symplectic mfd

& $$T^*(X/G) \simeq \mu^{-1}(0)/G$$

as symplectic mfds

Assume X is nonsingular variety $/k$

G reductive gp, $\mu^{-1}(0)$ is affine so

$$\mathcal{M}_0 = (\mu^{-1}(0))/\!/G \quad \& \quad \text{more generally}$$

$$\mathcal{M}_\chi = (\mu^{-1}(0))/\!/_\chi G$$

Ex) $X = k^2$, $G = k^*$

$$T^*X = \{(i,j) \mid i : k^2 \to k,\ j : k \to k^2\}$$

$$\Rightarrow \mu(i,j) = ij$$

$$\Rightarrow \mu^{-1}(0) = \{(i,j) \mid ij = 0\}$$

Quiver Varieties

Rep. Space

Why GIT? Often $R(v)/GL(v)$ are usually non-Hausdorff

For a quiver $\vec{Q}$ & dimension vector $v \in \mathbb{Z}_+^I$

$R(v) = \bigoplus \mathrm{Hom}(\mathbb{C}^{v_i}, \mathbb{C}^{v_j})$ & $GL(v)$ acts

by conjugation

Ex)

$$\vec{Q} = \begin{tikzcd} \underset{1}{\bullet} \arrow[r] & \underset{2}{\bullet} \end{tikzcd}$$

$$v = (1,1)$$

$$\begin{tikzcd} \overset{\mathbb{C}}{\bullet} \arrow[r] & \overset{\mathbb{C}}{\bullet} \end{tikzcd}$$

$$R(v) = \mathrm{Hom}(\mathbb{C}, \mathbb{C}) = \mathbb{C}$$

$$GL(v) = \prod GL(v_i, \mathbb{C})$$
$$= \mathbb{C}^\times \times \mathbb{C}^\times$$

$$(\lambda, \mu) \cdot x = \lambda x \mu^{-1}$$

As $\mathbb{C}^\times \subset GL(v)$ acts trivially we have an action of

$$PGL(v) = GL(v)/\mathbb{C}^\times$$

$$\left\{ \begin{array}{l} \text{Points of} \\ R(v)/PGL(v) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{iso classes} \\ \text{of reps of } \vec{Q} \\ \text{of dim } v \end{array} \right\}$$

Thm: There is a bijection between $\mathcal{R}_0(v) = R(v)//PGL(v)$ & iso classes of semisimple reps of $\vec{Q}$

Let $\theta \in \mathbb{Z}^I$ define a character

$$\chi_\theta : GL(v) \longrightarrow \mathbb{C}^\times$$
$$g \longmapsto \prod \det(g_i)^{-\theta_i}$$

θ is the stability parameter

For this character to be well defined we must have

$$\theta \cdot v = \sum \theta_i v_i = 0$$

then $\mathcal{R}_\theta(v) = R(v) /\!/_{\chi_\theta} PGL(v)$

<u>**Def:**</u> Let $\theta^I \in \mathbb{R}^I$. A rep V of $\vec{Q}$ is called θ-semistable (resp. θ-stable) if $\theta \cdot \dim V = 0$ & for any subrep $V' \subset V$ $\theta \cdot \dim V' \leq 0$ (resp. for every nonzero proper subrep V' we have $\theta \cdot \dim V' < 0$)

<u>**Thm:**</u> Let $v \in \mathbb{Z}_+^I$, $\theta \in \mathbb{Z}^I$ st $\theta \cdot v = 0$

Then an element $x \in \mathcal{R}(v)$ is χ_θ-ss iff V^x (corresponding rep of $\vec{Q}$) is θ-ss

Let Q be a graph, $Q^\#$ the double quiver

$Q = \bullet \longrightarrow \bullet \qquad Q^\# = \bullet \rightleftarrows \bullet$

$$R(Q^\#, v) = T^*(R(\vec{Q}, v))$$

We get a symplectic form ω_Ω on $R(Q^\#, v)$

$$\omega_\Omega(x_1 + y_1, x_2 + y_2) = \langle y_1, x_2 \rangle - \langle y_2, x_1 \rangle$$

$$x_i \in R(\vec{Q}, v)$$

$$y_i \in R(\vec{Q}, v)^*$$

Thm: The action of $GL(v)$ on $R(Q^\#, v)$ is Hamiltonian; the moment map

$$\mu_v : R(Q^\#, v) \longrightarrow \bigoplus \mathfrak{gl}(v_i, \mathbb{C})$$

is

$$z \longmapsto \bigoplus_i \sum_{t(h)=i} \varepsilon(h) z_h z_{\bar{h}}$$

Ex) $Q = \bullet\!\!-\!\!\!-\!\!\bullet$, $v=(1,1)$

$$Q^{\#} = \bullet \rightleftarrows \bullet$$

$$\mu_v^{-1}(0) = \{(x,y)\in\mathbb{C}^2 \mid xy = yx = 0\}$$

& $\mathcal{M}_0(v) = \{pt\}$

$$\parallel$$

$$\mu_v^{-1}(0)/\!/PGL(v)$$

Ex) $\vec{Q} = \bullet \circlearrowleft$ $\qquad v=n$

$$\theta\cdot v = 0 \Rightarrow \theta = 0$$

In this case $R(\vec{Q},v) = End(\mathbb{C}^n)$

$$\begin{aligned} R_0(n) &= R(n)/\!/PGL(n) \\ &= Spec\left(\mathbb{C}[\lambda_1,\dots,\lambda_n]^{S_n}\right) \\ &= \mathbb{C}^n/S_n \cong \mathbb{C}^n \end{aligned}$$

Finally we have framings

Def: Let W be an I-graded vector space $W = \bigoplus_{k \in I} W_k$. A W-framed rep of $\vec{G}$ is a rep $V = (V_k, x_h)$ of $\vec{G}$ together w/ a collection of linear maps

$$j_k : V_k \to W_k$$

$$\begin{tikzcd}
W_1 & W_2 & W_3 \\
V_1 \arrow[u, "j_1"] \arrow[r] & V_2 \arrow[u, "j_2"] \arrow[r] & V_3 \arrow[u, "j_3"]
\end{tikzcd}$$

$$R(V,W) = \left(\bigoplus_{k \in I} \mathrm{Hom}_{\mathbb{C}}(V_k, W_k) \right) \oplus \left(\bigoplus_{h \in \Omega} \mathrm{Hom}_{\mathbb{C}}(V_{s(h)}, V_{t(h)}) \right)$$

Ex) Let $\theta > 0$, $\vec{Q} = \bullet$, then reps of $\vec{Q}$ are vector spaces, and for $\dim V = n$, $\dim W = k$

$$R_\theta(n,k) = \{ j : \mathbb{C}^n \to \mathbb{C}^k \mid \ker j = \{0\} \} / GL(n,\mathbb{C})$$
$$= Gr(n,k)$$

Ex) Let $\theta > 0$, $\vec{Q}$ type A-quiver

$$\vec{Q} = \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots \longrightarrow \bullet$$

$$w = (0, 0, \dots, 0, r)$$

$$R^{ss}(v,w) = \{ (x_1, \dots, x_{\ell-1}, j) \}$$

where $x_i : V_i \to V_{i+1}$
$j : V_\ell \to \mathbb{C}^r$

$$R_\theta(v,w) = \mathcal{F}(v_1, v_2, \dots, v_\ell, r)$$

Framed Reps of Double Quivers

$$R(Q^{\#},V,W) = R(Q^{\#},V) \oplus L(V,W) \oplus L(W,V)$$

$$R(Q^{\#},V) \to \bigoplus \mathrm{Hom}(V_i,V_j) \qquad L(V,W) = \bigoplus \mathrm{Hom}(V_i,W_i) \qquad L(W,V) = \bigoplus \mathrm{Hom}(W_i,V_i)$$

<u>Note</u>: $R(Q,v)$ is often empty, hence why framings are used

$$\begin{tikzcd}
\bullet \arrow[r, shift left] \arrow[d, shift right] & \bullet \arrow[l, shift left] \arrow[d, shift right] \arrow[r, shift right] & \bullet \arrow[l, shift right] \arrow[d, shift right] \\
\bullet \arrow[u, shift right] & \bullet \arrow[u, shift right] & \bullet \arrow[u, shift right]
\end{tikzcd}$$

For each choice of orientation $\vec{Q} = (\Phi, \Omega)$

$$\mathrm{Rep}(Q^{\#},v,w) = T^{*}\mathrm{Rep}(\vec{Q},v,w)$$

Every choice of orientation gives rise to a symplectic structure on $\mathrm{Rep}(Q^{\#},v,w)$

Doubling a quiver does too

Every choice of skew-sym functn $\varepsilon : H \to \mathbb{C}^\times$ gives rise to a symplectic form on $\mathrm{Rep}(Q^\#, v, w)$

$$\omega_\varepsilon(z,i,j,z',i',j') = \mathrm{tr}_V\left(\sum_{h \in H} \varepsilon(h) z_{\bar{h}} z'_h + \sum_{k \in \bar{I}} i_k j'_k - i'_k j_k\right)$$

There is a natural action of

$$GL(v) \curvearrowright \mathrm{Rep}(Q^\#, v, w) \text{ by}$$

$$g(z,i,j) = (gzg^{-1}, gi, jg^{-1})$$

The moment map is

$$\mu_{v,w}(z,i,j) = \sum_{h \in H} \varepsilon(h) z_{\bar{h}} z'_h - \sum_k i_k j_k$$

Def: $\varepsilon : H \to \mathbb{C}^\times$ skew sym function

$\chi = \chi_\theta$, $\theta \in \mathbb{Z}^I$ the variety

$$\mathcal{M}_\theta(v,w) = \mu^{-1}(0) /\!/_\chi GL(v)$$

is the quiver variety

$\mathcal{M}_\theta(v,w)$ is quasiprojective w/ proj morphism

$$\mathcal{M}_\theta(v,w) \longrightarrow \mathcal{M}_0(v,w)$$

Both have a Poisson structure

§10.5 talks Stability

Thm 10.35/36 is important

Says $\mathcal{M}_\theta$ has certain dimension, non-deg Poisson structure

$\mathcal{M}_\theta$ Connected

Thm $\theta \in \mathbb{Z}^I$ be v-generic, assume $\mathcal{M}_0^{reg}(v,w)$ nonempty

then $\pi : \mathcal{M}_\theta(v,w) \longrightarrow \mathcal{M}_0(v,w)$ is a symplectic resolution of singularities

Ex) Type A quivers & flag varieties

Let $\theta > 0$,

$$\vec{Q} = \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots \bullet \longrightarrow \bullet \longrightarrow \bullet$$

(ℓ vertices

$$\vec{w} = (0, \dots, 0, r)$$

i.e.

$$\begin{tikzcd}
Q^{\#} = \bullet \arrow[r, shift right, "x_1"'] & \bullet \arrow[l, shift right, "y_1"'] \arrow[r, shift right, "x_2"'] & \bullet \arrow[l, shift right, "y_2"'] \arrow[r, shift right, "x_3"'] & \cdots \bullet \arrow[l, shift right, "y_3"'] \arrow[r, shift right, "x_{\ell-2}"'] & \bullet \arrow[l, shift right, "y_{\ell-2}"'] \arrow[r, shift right, "x_{\ell-1}"'] & \bullet \arrow[l, shift right, "y_{\ell-1}"'] \arrow[d, shift left, "j"] \\
& & & & & \bullet\ \mathbb{C}^r \arrow[u, shift left, "i"]
\end{tikzcd}$$

$$R^{ss}_{ep}(Q^{\#}, v, w) = \{ (x_1, \dots, x_{\ell-1}, y_1, \dots, y_{\ell-1}, i, j) \}$$

Any element $(x, j) \in R^{ss}$ defines a flag

$$V_1 \subset V_2 \subset \cdots \subset V_\ell \subset \mathbb{C}^r$$

$\mu(x,y,i,j) = 0$ gives

$$y_i|_{V_i} = y_{i-1} \qquad i|_{V_1} = y_{\ell-1}$$

$$\iff \qquad y_k = i|_{V_{k+1}}$$

So

$$\mathcal{M}(v,w) = \{(F,y)\}, \quad F = (0 \subset V_1 \subset V_2 \subset \cdots \subset V_\ell \subset \mathbb{C}^r)$$

– a partial flag in $\mathbb{C}^r$

$$y : \mathbb{C}^r \to \mathbb{C}^r : \; y(V_i) \subset V_{i-1}$$

$\mathcal{M}(v,w)$ can be identified with a cotangent bundle

$$\mathcal{F} = GL(r,\mathbb{C}) / P(v)$$

$$\underbrace{P(v)}_{\text{Parabolic subgroup}} = \{A \in GL(r,\mathbb{C}) : AV_i \subset V_i\}$$

i.e. the stabilizer

<u>Thm</u>: $Q = A_\ell$, $\theta > 0$

1) $\mathcal{M}(v,w) = T^* \mathrm{Rep}_\theta(v,w)$

$\mathrm{Rep}_\theta = \mathcal{F}(v,w)$ the flag variety

2) $\mu_w(x,y,i,j) = ji$

Moment map for action of $GL(w)$

<u>Ex</u> If $Q = A_1$, $\mathcal{F}(n,r)$ is the Grassmannian, so

$$\mathcal{M}(n,r) = T^* Gr(n,r)$$

$$\mathcal{M}_0(n,r) = \{ y \in \mathrm{End}(\mathbb{C}^r) \mid y^2 = 0,\ \mathrm{rank}(y) \le n \}$$

We can describe $\mathcal{M}(n,r)$ as pairs (y, V)

$V \subset \mathbb{C}^r$ $\dim V = n$

$y \in \mathrm{End}(\mathbb{C}^r)$, $\mathrm{Im}(y) \subset V$, $y(V) = 0$

Chpt 11

Let X be affine variety over $\mathbb{C}$

$$S^n X = X^n /\!/ S_n = \operatorname{Spec}\left(\mathbb{C}[X]^{\otimes n}\right)^{S_n}$$

↳ symmetric powers

$$S^n \mathbb{C} \simeq \mathbb{C}^n$$

<u>Def:</u> $\operatorname{Hilb}^n X = \left\{ J \subset \mathbb{C}[X] \;\middle|\; \begin{array}{l} J \text{ is an ideal in } \mathbb{C}[X] \\ \dim\left(\mathbb{C}[X]/J\right) = n \end{array} \right\}$

as a set

It's also a scheme (what is $\mathcal{O}_X$? What is it the Spec of?)

Alternatively, we have $\operatorname{Hilb}^n X$ is the set of iso classes of pairs (M, v)

M - $\mathbb{C}[X]$ module of dim n

v cyclic vector in M

Thm:

1) Hilbert-Chow morphism

$$\pi : \mathrm{Hilb}^n X \longrightarrow S^n X$$
$$J \longmapsto \mathrm{Supp}(\mathbb{C}[x]/J)$$

Thm: X nonsingular of dim 2, $\mathrm{Hilb}^n X$ is smooth & $\pi : \mathrm{Hilb}^n X \to S^n X$ is a resolution of singularities

Ex) $X = \mathbb{C}^2$

$t = (t_1, t_2) \in S^2_\circ X$ i.e. $t_1 \neq t_2$

$\pi^{-1}(t) = J_t$ is a single pt

To study $\pi^{-1}(t)$ for $t = (t, t)$ consider $t = (0,0)$

$M = \mathbb{C}[x]/J$, $J \in \pi^{-1}(t)$, this is a 2-dim module over $\mathbb{C}[x] = \mathbb{C}[z_1, z_2]$ generated by a single vector & acts nilpotently on z_1, z_2 so

$$M = \mathbb{C}[z_1, z_2] / (z_1^2, z_2^2, z_1 z_2, \alpha z_1 + \beta z_2)$$

So $\pi^{-1}(0,0) \simeq \mathbb{P}^1$

Similarly $\pi^{-1}(t) = \mathbb{P}^1$

$Q = \bullet \circlearrowleft$

$$Rep_0(n) \simeq \mathbb{C}^n$$
$$M_0(n) = \mathbb{C}^{2n}/S_n$$

Consider

$$\begin{tikzcd} \overset{V}{\bullet} \arrow[loop left, "x"] \arrow[loop right, "y"] \arrow[d, shift right, "j"'] \\ W \arrow[u, shift right, "i"'] \end{tikzcd}$$

Thm: $M_\theta(n,1) \simeq \mathrm{Hilb}^n \mathbb{C}^2$

$\pi : M_\theta(n,1) \to M_0(n,1)$

is the Hilbert-Chow morphism

$$\mu^{-1}_{n,1}(0) = \left\{ x,y : \mathbb{C}^n \to \mathbb{C}^n,\ \begin{array}{l} i : \mathbb{C} \to \mathbb{C}^n \\ j : \mathbb{C}^n \to \mathbb{C} \end{array} \;\middle|\; [x,y] - ij = 0 \right\}$$

$$\mathcal{M}_\Theta(n,1) = \left(\mu^{-1}(0)^s / GL_n(\mathbb{C}) \right) = \mathrm{Hilb}^n \mathbb{C}^2$$

<u>Thm</u>: $\mathrm{Hilb}^n \mathbb{C}^2$ is symplectic, hyperkähler mfd

Moduli Space of torsion free sheaves..

<u>Def</u>: A quasicoherent sheaf is called torsion free if $\forall\, U \subset X$ affine open, $\mathcal{F}(U)$ is torsion free as a module over $\mathcal{O}(U)$:

$$\forall \text{ nonzero section } s \in \mathcal{F}(U),\ f \in \mathcal{O}(U)$$
$$fs \neq 0$$

<u>Ex</u>) If V is locally free (sheaf of sections of a vector bundle), any subsheaf is locally free

For any quasicoherent sheaf $\mathcal{F}$, $\mathcal{F}^\vee := \mathcal{H}om(\mathcal{F}, \mathcal{O})$

Thm: X nonsingular variety, $\mathcal{F}$ coherent torsion free sheaf on X:

1) $\exists$ Zariski open $U \subset X$ codim ≥ 2 st $\mathcal{F}|_U$ is locally free

2) If $\dim X = 2$, $\mathcal{F}^{\vee\vee}$ is locally free of finite rank & $\mathcal{F} \to \mathcal{F}^{\vee\vee}$ is injective

$$\mathcal{F}|_U \simeq \mathcal{F}^{\vee\vee}|_U$$

So X smooth of dim 2, any coherent torsion free sheaf on X is iso over an open dense subset $U \subset X$ to the sheaf of sections of a vector bundle.

As any coherent sheaf admits a resolution by vector bundles, we can define Chern classes

$$c_i(\mathcal{F}) \in H^{2i}(X)$$

Ex) $J \in \mathrm{Hilb}^n X$, $\mathcal{F}_J$ corresponding subsheaf of $\mathcal{O}$ so $\Gamma(X, \mathcal{F}_J) = J$

then $\mathcal{F}_J$ is torsion free &

$\therefore$ $\Gamma(X, \mathcal{O}/\mathcal{F}_J) = \mathbb{C}[x]/J$

Ex) $f: \mathcal{O} \to \mathcal{O} \oplus \mathcal{O}$

$s \mapsto (z_1 s, z_2 s)$

$\mathrm{Im}\, f = \{(s_1, s_2) \in \mathcal{O} \oplus \mathcal{O} \mid z_2 s_1 = z_1 s_2\}$

$\mathcal{F} = \mathcal{O} \oplus \mathcal{O} / \mathrm{Im}\, f$ is torsion free

$g: \mathcal{O} \oplus \mathcal{O} \to \mathcal{O}$

$(s_1, s_2) \mapsto z_2 s_1 - z_1 s_2$

$\Rightarrow$ $\mathrm{Ker}(g) = \mathrm{Im}(f)$

Lemma: V be $\dim < \infty$ vec. space $/ \mathbb{C}$

$A_1, A_2 : V \to V$ operators

$\mathcal{V} = V \otimes \mathcal{O}$ sheaf of V-valued functions on $\mathbb{C}^2$

$$A : \mathcal{V} \to \mathcal{V} \oplus \mathcal{V}$$
$$v \longmapsto \big((A_1 - z_1)v, (A_2 - z_2)v\big)$$

A is injective & $\mathcal{V} \oplus \mathcal{V} / \operatorname{Im}(A)$ is torsion free

Let $l_\infty = \{(0 : z_1 : z_2)\} \subset \mathbb{P}^2$

$$\mathbb{P}^2 \setminus l_\infty \simeq \mathbb{C}^2$$

Def: Let $\mathcal{F}$ be a torsion free sheaf of rank r on $\mathbb{P}^2$. A framing is an isomorphism

$$\phi : \mathcal{F}|_{l_\infty} \xrightarrow{\sim} \mathcal{O}_{l_\infty}^{\oplus r}$$

$\mathcal{M}^{fr}(n,r) = \{ \text{iso classes of pairs } (\mathcal{F}, \phi) \}$

$\mathcal{F}$: torsion free sheaf of rank r on $\mathbb{P}^2$

$c_2(\mathcal{F}) = n$

ϕ - framing

As a set, this also has scheme structure & a fine moduli space

Ex) $r=1$, $c_1(\mathcal{F})=0 = c_1(\mathcal{F}^{\vee\vee})=0$

$\Rightarrow \mathcal{F}^{\vee\vee} \simeq \mathcal{O}$

As $\mathcal{F} \hookrightarrow \mathcal{F}^{\vee\vee} = \mathcal{O}$

$\mathcal{O}/\mathcal{F}$ is coherent on $\mathbb{P}^2$ which is zero in a nbhd of l_∞

$\dim \Gamma(\mathbb{C}^2, \mathcal{O}/\mathcal{F}) = n$

So $M = \Gamma(\mathbb{C}^2, \mathcal{O}/\mathcal{F})$ is an n-dim module over $\mathbb{C}[\mathbb{C}^2] = \mathbb{C}[z_1, z_2]$

$\mathcal{M}^{fr}(n,1) \simeq \mathrm{Hilb}^n \mathbb{C}^2$

In general

$$\mathcal{M}^{fr}(n,r) \simeq \mathcal{M}_\Theta(n,r) \quad \Theta < 0$$

§11.4 ASDC

Let X - Riemannian mfd.

$$\begin{array}{c} E \\ \downarrow \\ X \end{array}$$

Complex vect. bundle of rank r over X w/ Hermitian metric

Let $\mathcal{A}$ be the space of metric connections on E; for a connection $A \longrightarrow F_A$ is the curvature