Outline

- Def of sheaf of $\mathcal{O}$-modules
- Def of vector bundle (fiber, sections, etc)
  - Examples (trivial bundle, tangent bundle, canonical line bundle)
- Def of holomorphic vector bundle
  - Examples
- Relation of Locally free sheaves & holomorphic vec. bundles
- Relation of Holomorphic connections & $\mathbb{C}$-local systems
  $\left(f' = \frac{f}{2z}\right)$

$\mathcal{O}(U) = \{f : U \to \mathbb{C} \text{ hol}\}$

Consider the ring sheaf $\mathcal{O}$ of holomorphic functions on our open domain $D \subset \mathbb{C}$ (connected open)

Def: A sheaf of $\mathcal{O}$-modules is a sheaf of abelian groups $\mathcal{F}$ on X st for each open $U \subset X$ $\mathcal{F}(U)$ has an $\mathcal{O}(U)$ module structure:

$$\mathcal{O}(U) \times \mathcal{F}(U) \longrightarrow \mathcal{F}(U)$$

st if $U \subset V$ the diagram commutes:

$$\begin{tikzcd}
\mathcal{O}(V) \times \mathcal{F}(V) \arrow[r, "\text{action}"] \arrow[d, "\mathrm{res}_{V,U} \times \mathrm{res}_{V,U}"'] & \mathcal{F}(V) \arrow[d, "\mathrm{res}_{V,U}"] \\
\mathcal{O}(U) \times \mathcal{F}(U) \arrow[r, "\text{action}"'] & \mathcal{F}(U)
\end{tikzcd}$$

Def: We say that $\mathcal{E}$ is locally free if every point of D there is an open nbhd $V \subset D$ st $\mathcal{E}|_V \cong \mathcal{O}^n|_V$ for some $n > 0$. n is called the rank & we say $\mathcal{E}$ is free of rank n if $\exists$ iso $\cong \mathcal{O}^n$ on all D

There is a relationship between locally free sheaves & holomorphic vector bundles

Should talk about this now

Complex

Def: A real vector bundle of complex dimension n over a topological space B consists of a topological space E & projection map (cts) $p : E \to B$ together with

Complex

the structure of a real vector space in each fiber $p^{-1}(b)$, $b \in B$ subject to the following local triviality:

Each point of B must possess a nbhd U so that the inverse image $p^{-1}(U)$ is homeomorphic to $U \times \mathbb{R}^n$ under a homeo that sends $p^{-1}(b) \to b \times \mathbb{C}^n$

$\mathbb{C}^n$

Ex The trivial bundle, $E = B \times \mathbb{R}^n$ & π is projection onto the first factor

e.g. $\underbrace{S^1 \times \mathbb{R}}_{\text{cylinder}}$ over S^1 is the trivial bundle

Note: Since $\mathbb{R}$ has real dim 1 this is an example of a line bundle

Ex The tangent bundle of a smooth mfd M

$$TM = \{ (x,v) \in M \times \mathbb{R}^n \mid \underbrace{v \text{ tangent to } x \text{ at } M}_{v \in T_xM} \}$$

All derivations at x

$D: C^\infty(M) \to \mathbb{R}$ linear

$D(fg) = D(f)g(x) + f(x)D(g)$

$$p: TM \to M$$
$$(x,v) \mapsto x$$

Let (U, ϕ_U) be a chart of M w/ coordinates $x^1, \dots, x^m$ it defines a trivialization

$$\psi_U : \pi^{-1}(U) \to U \times \mathbb{R}^m \text{ by}$$

$$\left(p, \sum_{k=1}^{m} a_k \frac{\partial}{\partial x^k}\Big|_p\right) \mapsto (p, (a_1, \dots, a_m))$$

Ex $\mathbb{C}P^n = (\mathbb{C}^n \setminus \{0\}) / \mathbb{C}^\times$

The tautological line bundle $p: E \to \mathbb{C}P^n$

$$E = \{(\ell, v) \mid v \in \ell\} \subset \mathbb{CP}^n \times \mathbb{C}^{n+1}$$

$$p(\ell, v) = \ell$$

$$\phi : p^{-1}(U) \to U \times X$$

$$\phi(\ell, v) = (\ell, p(v))$$

<u>Def</u>: A holomorphic vector bundle on $D \subset \mathbb{C}$ is one in which

$p : E \to D$ is holomorphic

and $p^{-1}(V) \cong V \times \mathbb{C}^n$ as complex manifolds inducing vector space isomorphisms on fibers

Given a vector bundle $p : E \to B$, a section is a map $s : B \to E$ assigning to each $b \in B$ a vector $s(b)$ in the fiber $p^{-1}(b)$

$\Leftrightarrow$ $ps = id$

Given a <u>holomorphic</u> vector bundle $p : \mathcal{E} \to B$ its sheaf of holomorphic sections $\mathcal{H}_{\mathcal{E}}$ is given as

$U \subset D$ $\quad \mathcal{H}_{\mathcal{E}}(U) = \{ s : U \to \mathcal{E} \mid p \circ s = id_U \}$

$$\mathcal{H}_\varepsilon|_V \cong \mathcal{O}^n|_V$$

So this is a locally free sheaf. This construction induces an equivalence of categories

$$\left\{ \begin{array}{c} \text{Category of} \\ \text{holomorphic vector} \\ \text{bundles on } D \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Locally free sheaves} \\ \text{on } D \end{array} \right\}$$

$V \subset D$ open st $p^{-1}(V) \cong V \times \mathbb{C}^n$

$$\mathcal{H}_\varepsilon|_V \cong \mathcal{O}^n|_V$$

Other direction: Sending $\mathcal{E}|_V$ to $V \times \mathbb{C}^n$ the patching together

Intuition: Locally constant sheaves on D are just holomorphic vector bundles on D viewed from below

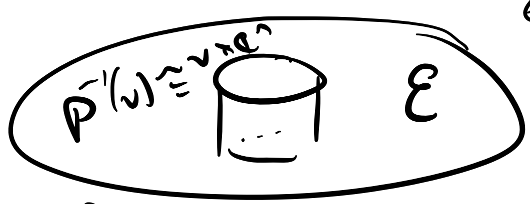
Top down

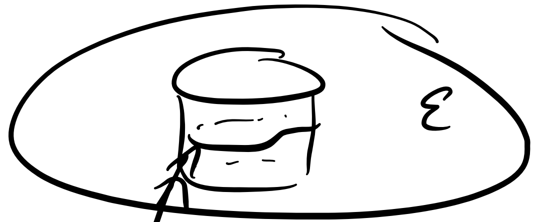
Holomorphic vector bundle

Bottom up

Locally free sheaf

Complex mfd

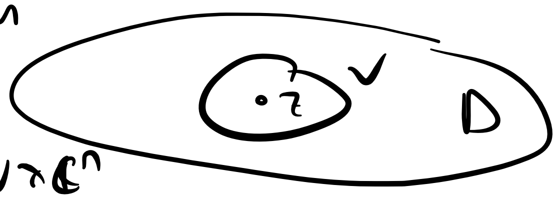




$\forall z \in D$ the fiber $p^{-1}(z)$ has the structure of $\mathbb{C}$-vector space $\mathcal{E}$

$\downarrow p$ ← hol. surjection

$\exists V \subset D$ open $z \in V$ st



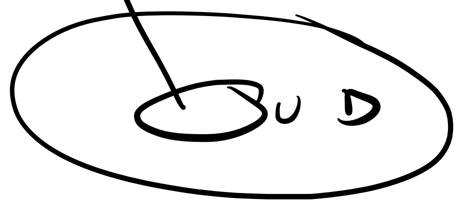
$D \subset \mathbb{C}$ connected open

$p^{-1}(V) \cong V \times \mathbb{C}^n$

S hol. sections

$s : U \to \mathcal{E}$

$p \circ s = id_U$



assigning a point in U to a vector in $\mathbb{C}^n$

V is like a indexing set

$\mathcal{E}$ $\longmapsto$ $\mathcal{H}_\mathcal{E}$

holomorphic vector bundle

sheaf of holomorphic sections

$\mathcal{H}_\mathcal{E}|_V \cong \mathcal{O}^n|_V$

Section of hol. vect. bundle is vector field on U

pick open set small enough like V the preimage is $V \times \mathbb{C}^n$ so sections are maps $V \to \mathbb{C}^n$ (n tuples of hol functions)

We also need the sheaf Ω^1_D of hol 1-forms on D: Given hol f on some $U \subset D$

$df : U \to \mathbb{C}$ by $z \mapsto f'(z)$

A section of $\Omega^1_D(U) = \{ \omega_U : U \to \mathbb{C} \mid \forall z \in U \ \exists$ an nbhd $V \subset U$ ...

isomorphically onto
an open disc
around 0 in $\mathbb{C}$
by a hol. function
$g \in \mathcal{O}(V)$ st
$\omega|_V = f\,dg$
$f \in \mathcal{O}(V)$

This is a sheaf of $\mathcal{O}$-modules on D.

Note: $dz : D \to \mathbb{C}$ is a global section of Ω^1_D $\left(d(z-a) = dz \;\; \forall a \in D \right)$
and so is df for arbitrary $f \in \mathcal{O}(D)$
since $df = f'dz$, in particular we can
identify $\frac{df}{dz} : D \to \mathbb{C}$ w/ $f' \in \mathcal{O}(V)$

Huenhuen: Connections & local systems

If time:

Ex] Let's look at an ODE w/ singularities

$$f' = \frac{f}{2z} \quad \text{on} \quad D = \mathbb{C} \setminus \{0\}$$

This is a first order ODE, so what is the holomorphic connection? $(\mathcal{O}_D, \nabla)$

∇ - connection corresponding to $f' = f/2z$

$\forall f \in \mathcal{O}_D(U)$, $U \subset D$ open

Note: This looks like the differential equation

$$\nabla(f) = df - \underbrace{\left(\frac{f}{2z}\right)dz}_{\text{1-form on } D}$$

$\nabla : \{\text{sections of } \mathcal{O}_D\} \longrightarrow \{\text{1-forms}\}$

Holomorphic connection $\longrightarrow$ $\mathbb{C}$-local system:

$(\mathcal{O}_D, \nabla) \longmapsto \mathcal{O}_D^{\nabla}$ where

$\mathcal{O}_D^{\nabla}$ - subsheaf $(\mathcal{O}_D^{\nabla} \subset \mathcal{O}_D)$ of horizontal sections

$\hookrightarrow$ $f \in \mathcal{O}_D(U)$ st $\nabla(f) = 0$

$\hookrightarrow$ $\nabla(f) = 0 \iff df = \frac{f}{2z}dz$ (equality of 1-forms)

$df = f'dz$

$= \frac{f}{2z}dz$

$\Rightarrow$ coefficient functions must be equal

$\iff f' = \frac{f}{2z}$

$\hookrightarrow$ i.e. f is a "local solution" to the ODE

[If $\mathcal{O}_D$ is a free sheaf, the holomorphic connection can be represented as an ODE on the entire domain, but if locally free only, then only local solutions

In this example,

$$\mathcal{O}_D^{\nabla}(U) = \{ c \cdot \sqrt{z} \mid c \in \mathbb{C} \}$$

$n=1$ dim vector vector space

Ex) Correspondence for trivial bundle & sheaf — Exterior derivative is a flat connection on trivial line bundle

— - - - tangent bundle & sheaf — Covariant deriv

Def: A holomorphic connection ∇ on a holomorphic vector bundle $p: E \to D$ is a $\mathbb{C}$-linear homomorphism of sheaves

$$\nabla : \mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_D} \Omega^1_D$$

Satisfying the Leibniz rule:

$\forall$ U open in D, section s, $f \in \mathcal{O}_D(U)$

$$\nabla(fs) = df \otimes s + f\nabla(s)$$

the tensor product is defined as

$$(\mathcal{E} \otimes_{\mathcal{O}} \Omega^1_D)(U) = \mathcal{E}(U) \otimes_{\mathcal{O}(U)} \Omega^1_D(U)$$

In our example, $f' = f|_{\mathbb{Z}_7}$, $\mathcal{E} = \mathcal{O}_D$ is the (locally) free sheaf on D

So

$$\mathcal{E} \otimes_{\mathcal{O}} \Omega^1_D = \mathcal{O}_D \otimes_{\mathcal{O}} \Omega^1_D$$
$$\simeq \Omega^1_D \quad (\text{Eval on } U)$$

Note: For any A-module M, $A \otimes_A M = M$

In general, $\mathcal{E}$ is free of rank n, $\mathcal{E} \simeq \mathcal{O}_D^n$

$$\mathcal{E} \otimes_{\mathcal{O}} \Omega^1_D \simeq (\Omega^1_D)^{\oplus n}$$

In general, $\mathcal{E}$ is any locally free of rank n, so

$\mathcal{E}|_V \simeq \mathcal{O}_D^n|_V$ locally, and restricting

to any small open set V,

$$(\mathcal{E} \otimes_{\mathcal{O}} \Omega^1_D)|_V \simeq \mathcal{E}|_V \otimes_{\mathcal{O}|_V} \Omega^1_D|_V$$
$$\simeq \mathcal{O}_D^n|_V \otimes_{\mathcal{O}|_V} \Omega^1_D|_V$$
$$\simeq \mathcal{O}_D^n \otimes \ldots \Omega^1 \ldots$$

$\cong (\Omega^1_V)^{\oplus n}$

$\left\{\begin{array}{l}\text{flat sections}\\ \text{of complex vect. bund}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{category of}\\ \text{local system}\end{array}\right\}$

But this is false for algebraic flat vect. bundles

On $\mathcal{O}_{\mathbb{A}^1}$

$$\left.\begin{array}{l}\nabla_1(f) = df\\ \nabla_2(f) = df - f\,dz\end{array}\right\} \text{Equivalent as analytic connections}$$

They both have the same sheaf of flat sections, constant sheaf

But not equiv as algebraic connections

∇_1 has flat alg sect.

∇_2 doesn't, $f' - f = 0$ has irregular sing at ∞

X - vector field

$$\begin{tikzcd} \mathcal{E} \arrow[r] \arrow[rr, bend left] & \mathcal{E} \otimes \Omega^1 \arrow[r, "X"] & \mathcal{E} \end{tikzcd}$$

$\nabla_X s$ a section $\mathcal{E}$

$$\nabla_X (fs) = X(f)s + f\nabla_X s$$