Outline

1) Describe $\mathrm{Hilb}^n \mathbb{C}^2$ as a set
2) Describe $\mathrm{Hilb}^n \mathbb{C}^2$ as GIT quotient (quiver variety)
3) Ordinary & Equivariant Cohomology
4) Fixed pts

Blue: Talk   Black: write

The Hilbert Scheme of n points in $\mathbb{C}^2$ is the configuration space of n-tuples of points in $\mathbb{C}^2$:

$P = \{p_1, \ldots, p_n\}$ unordered pts in $\mathbb{C}^2$

is uniquely specified by

$$I_P = \{ f(p_1) = \cdots = f(p_n) = 0 \} \subset \mathbb{C}[\mathbb{C}^2]$$

$$\shortparallel$$

$$\mathbb{C}[x_1, x_2]$$

coord. ring $\mathbb{C}^2$

$$\mathcal{O}_P = \text{functions on } P = \mathbb{C}[x_1, x_2] / I_P$$

$$\dim \mathcal{O}_P = n$$

$$\mathrm{Hilb}^n(\mathbb{C}^2) = \{ \text{Ideals } I \subset \mathbb{C}[x_1,x_2] \mid \mathbb{C}[x_1,x_2]/I = n \}$$

This describes $\mathrm{Hilb}^n \mathbb{C}^2$ as a set. There is a natural scheme structure associated to it. [FGA Explained Chpt 7]

Facts of $\mathrm{Hilb}^n(\mathbb{C}^2)$:

- Nonsingular (i.e. smooth)
- Irreducible
- Quasiprojective
- Symplectic
- $\dim = 2n$

$I_p \mapsto p$ extends to a map

$$\pi : \mathrm{Hilb}^n \mathbb{C}^2 \to \mathrm{Sym}(\mathbb{C}^2)$$

that is proper & birational

This makes $\mathrm{Hilb}^n \mathbb{C}^2$ an equivariant symplectic resolution

As mentioned there are some interesting combinatorics at play: The action of scaling coordinates of $\mathbb{C}^2$ lifts to the Hilbert Scheme:

$(\mathbb{C}^\times)^2 = T \circlearrowright \mathbb{C}^2$

$$(t_1, t_2) \cdot (x_1, x_2) = (t_1 x_1, t_2 x_2)$$

t_1, t_2 are the equivariant parameters of T on Hilb

Clearly an ideal $I \in Hilb^n(\mathbb{C}^2)$ is fixed iff generated by monomials

$$(t_1, t_2) \cdot I = \{ f(t_1 x_1, t_2 x_2) \mid f(x_1, x_2) \in I \}$$

What is the weight of to preserve degree?

$$f(t_1 x_1, t_2 x_2) = t_1^a t_2^b f(x_1, x_2)$$

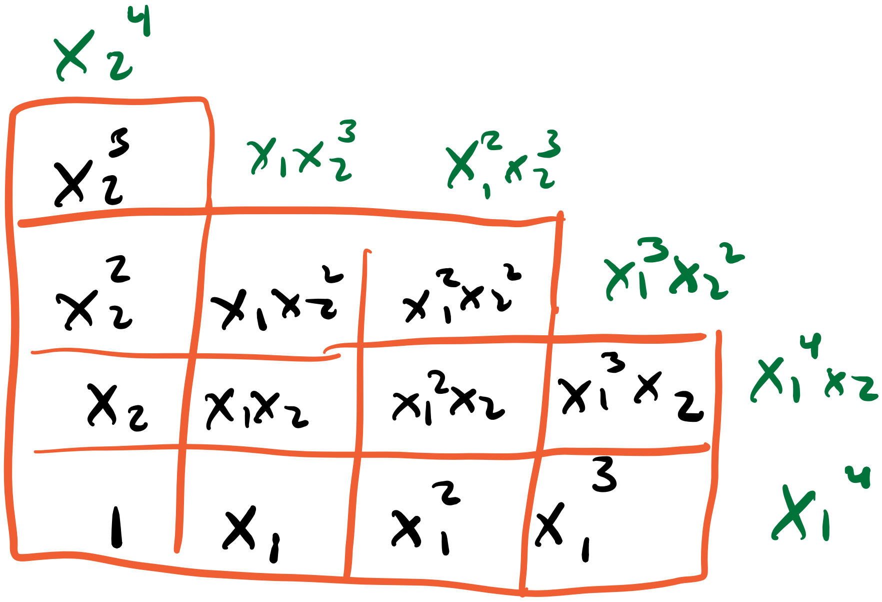
as obviously $f(x_1, x_2) \in I \Rightarrow$

$$t_1^a t_2^b f(x_1, x_2) \in I$$

so $f(x_1, x_2)$ must be monomials

$$x_1^{a_i} x_2^{b_i}$$

Monomial ideals correspond to Young Diagrams



A fixed pt of $\mathrm{Hilb}^n(\mathbb{C}^2)$

Correspondng to $x_2^4,\ x_1x_2^3,\ x_1^2x_2^3,\ x_1^3x_2^2,\ x_1^4x_2,\ x_1^4$

Philosophy: To be an ideal & a monomial in I you must choose everything below & to the left

Next, we'll describe $\mathrm{Hilb}^n(\mathbb{C}^2)$ as a quiver variety

Quick quiver variety refresher

Def: Q - graph, I - vertex set

$$\mathrm{Rep}_Q(v,w) = \bigoplus_{i\to j} \mathrm{Hom}(V_i,V_j) \oplus \bigoplus_{i\in I} \mathrm{Hom}(W_i,V_i)$$

$v, w \in \mathbb{Z}_{\geq 0}^{I}$

dimension framing

$G_v = \prod_{i \in I} GL(v_i)$ acts by change of basis

on $\mathrm{Rep}_Q(v,w)$

$$A_{ij} \mapsto g_j A_{ij} g_i^{-1}$$

$$A_{ij} \in \mathrm{Hom}(V_i, V_j)$$

$$(g_i, g_j) \in GL(V_i) \times GL(V_j) \subset G_v$$

$$I_i \in \mathrm{Hom}(W_i, V_i)$$

$$I_i \mapsto g_i I_i$$

This induces a Hamiltonian action on $T^*\mathrm{Rep}_Q(v,w)$

$$\mu : T^*\mathrm{Rep}_Q(v,w) \longrightarrow \mathrm{Lie}(G_v)^*$$

following 1 arrow then follows the opposite arrow back

Let $\theta : G_v \to \mathbb{C}^\times$ be a character

$$g \mapsto \prod \det g_i$$

$\mathcal{M}_{\theta,Q}(v,w) = \mu^{-1}(0)/\!/_\theta\, G_v$ is the

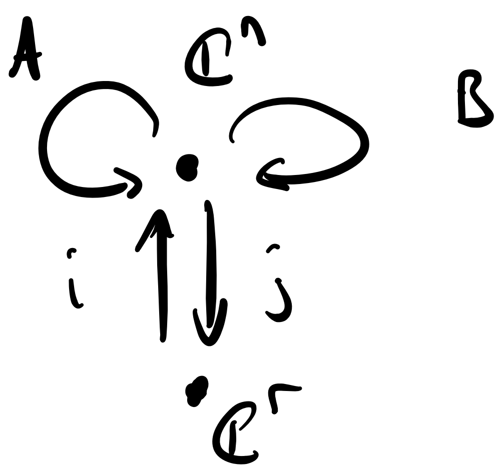
Nakajima's Quiver variety

$$\mu(A,B,I,J) = \sum_{j\to i} A_{j,i}B_{i,j} - \sum_{i\to j} B_{j,i}A_{i,j} - J_iI_i$$

$$T^*\mathrm{Rep}_Q(v,w) = \mathrm{Rep}(v,w) \oplus \mathrm{Rep}(v,w)^*$$

$Q =$ 

Doubling & adding a framing:



In this case $\mu(A,B,i,j) = [A,B] + ij$

$$\mathcal{M}(n,r) = \mu^{-1}(0) /\!/_\theta G_V$$

If $r=1$, i.e.

$$\begin{array}{c} G \curvearrowright \mathbb{C}^n \circlearrowleft \\ \uparrow\downarrow \\ \mathbb{C} \end{array}$$

$$\mathcal{M}(n,1) \simeq \mathrm{Hilb}^n \mathbb{C}^2$$

Note: The stability condition states there is no subspace $S \subset \mathbb{C}^n$ st $A(S) \subset S$, $B(S) \subset S$ & $\mathrm{Im}(i) \subset S$

Aside: $\mathcal{M}(n,r)$ is also known as two other things:

1) Moduli space of framed rank r torsion free sheaves $\mathcal{F}$ on $\mathbb{P}^2$ w/ fixed second Chern class $c_2(\mathcal{F}) = n$

A Framing is a choice of iso

$$\mathcal{F}|_{L_\infty} \xrightarrow{\sim} \mathcal{O}_{L_\infty}^{\oplus r}$$

$L_\infty = [0 : p_2 : p_3]$ line at ∞ on $\mathbb{P}^2$

Quasicoherent Sheaf is torsion free if $\forall$ affine open $U \subset \mathbb{P}^2$ the space of local sections $\mathcal{F}(U)$ is torsion free as a module over $\mathcal{O}(U)$

i.e. $\forall s \in \mathcal{F}(U)$ $s \neq 0$, $f \in \mathcal{O}(U)$ $f \neq 0$, $fs \neq 0$

2) Moduli Space of framed rank r a-instantons on $\mathbb{R}^4$

Let X be a 4-real dimension oriented Riemannian mfd, E a rank r complex vector bundle over X w/ Hermitian metric

If A is a connection on E, F_A - curvature

<u>Def</u>: w/ the above assumptions on X, E

A metric connection A on E is called anti-self dual (ASD) if $F_A^+ = 0$

$$F_A \in \Omega^2(X, \mathfrak{su}(E))$$

$$= \Omega^+ \oplus \Omega^-$$

$$*: \Omega^2(X) \longrightarrow \Omega^2(X)$$

$$*^2 = \mathrm{id}$$