Hi, I finally typed out a solution to T5 on Aug 2021 in prep for the January exam. Hope this clears up all confusion on the problem. It's also a lot cleaner than looking for $N(H)$, I was stupid to not use some basic group theory in solving the problem.

1 T5

a

First we inspect the degree of these coverings. The degree is the number of preimages of points, and since both Y, Z are path connected the degrees are the same for any point in the spaces. If we look at the base points both Y, Z have 3 preimages, hence the degree of the coverings is 3 for both spaces. Recall the degree is also defined as the index of the fundamental groups of Y, Z under the induced homomorphism of the covering space. By definition $Aut(V) = N(H)/H$ where $H = p_*(\pi_1(Y, q_2))$, we can replace Z for Y and q_2 with any preimage of q. Since the index of H is 3 we know that the normalizer being a larger subgroup than H must be either H itself or $\pi_1(X, q)$, this follows from Lagrange's theorem and the 3rd isomorphism theorem: $H < N(H) < \pi_1(X, q) = G$, so

$$3 = |G/H| = |G/N(H)||N(H)/H|$$

So $|N(H)/H| = 1$ or 3. If 1 then $N(H) = H$ so the automorphism group is trivial, if 3 then $N(H) = G$ and in particular $|Aut(V)| = |N(H)/H| = 3 \implies Aut(V) \cong \mathbb{Z}_3$

Now for our Y, Z spaces $Aut(Y) \cong 1$ and $Aut(Z) \cong \mathbb{Z}_3$. The reason being for any point in Y, say q_3, we cannot map it to any other point in Y under automorphism, no other point in Y has a b_3 loop attached to it, the other two base points q_1, q_2 have b_1, b_2 coming either in or out but not both so in the picture there is no loop, just a lift "upwards". Another way we could see this it to try to imagine rotating or reflecting this picture, there are no other possible symmetries attached to Y other than the trivial one. As for Z this has all possible symmetries, we can effectively rotate this space 120° after some manipulations of 'pulling' q_1, q_3 downwards to get a more symmetric looking graph. Another way is to just say that each of q_1, q_2, q_3 has a loop of 'a' and of 'b' (rather the preimages of a, b) coming in and out.

b

Y is not a regular covering space by the above argument, but Z is regular.

c

Y is path connected so the degrees (indexes) are all the same, and it's 3.